

Chapitre 6 – Taylor's series

This chapter explains the Taylor's series, which allow any function to be expressed with polynomials.

We can thus approach any function with polynomials, which have the advantage of being easily calculated.

I - Formula de Taylor-Young

Taylor Expansion are based on Taylor's formula.

this allows us to write $f(x)$ as a sum of a polynomial called $P_n(x)$ and a rest called $R_n(x)$: $f(x) = P_n(x) + R_n(x)$.

$P_n(x)$ is called the polynomial part of Taylor's formula.

The polynomial part is the same, only the rest changes.

We then consider a function f defined on an interval I and a natural number n .

To be able to apply this formula, the function f must be of class C_n over I , i.e. it must be n times differentiable on I and $f^{(n)}$ must be continuous.

We can then write that for any real a belonging to the interval I , we have:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o((x-a)^n)$$

We can also write this formula as a sum:

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!}(x-a)^k + o((x-a)^n)$$

Most of the time, we take $a = 0$, which will simplify the calculations.
All the usual DLs to be learned by heart are the DL in 0.

Example 1

Let's take a look at the simplest application example you can have.

We take $f(x) = e^x$

f is obviously of class C^∞ , so we can take $n = 4$ for example.

We are looking for the Taylor Expansion at the order 4 from f to 0 (so $a = 0$).

According to the Taylor-Young formula:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + o(x^4)$$

Or $f(x) = e^x$ donc $f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = e^x$

Donc $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = e^0 = 1$

Thus, in the vicinity of 0, by replacing the factorials:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$$

Attention cela n'est vrai qu'au voisinage de 0, c'est-à-dire pour x proche de 0.

II-Développements limités usuels

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + o(x^n)$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n+1}x^n}{n} + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \dots + (-1)^n x^n + o(x^n)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 \dots + x^n + o(x^n)$$